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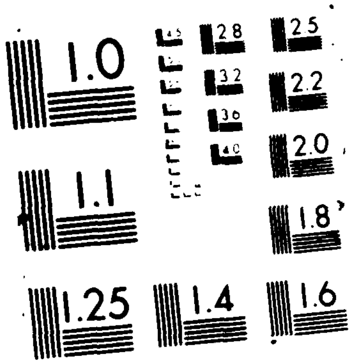
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We obtain a new sharp inequality for the Green's function of Brownian motion on a ball.




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# Green's Function for a Ball


K. L. Chung*


Let $B = B(a,r)$ be the open ball with center $a$ and radius $r$ in $R^d$, $d > 3$; $\partial B$ its boundary sphere. For $x \neq a$, its inversion with respect to $B$ is defined to be

(1) $$x^* = a + \frac{r^2}{|x - a|^2}(x - a).$$

We have

(2) $$|x - a||x^* - a| = r^2;$$

from which it follows that the mapping is involutary: $(x^*)^* = x$. Also we have from (1):

$$|x^* - y|^2 = |a - y|^2 + \frac{2r^2}{|x - a|^2}(a - y, x - a) + \frac{r^4}{|x - a|^4}|x - a|^2;$$

(3) $$|x^* - y|^2|x - a|^2 = |x - a|^2|y - a|^2 - 2r^2(x - a, y - a) + r^4.$$




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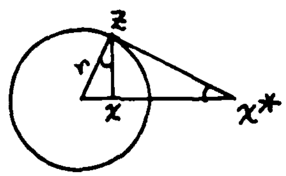
The right-hand member of (3) being symmetric in $(x,y)$, we see that

(4) $$|x^* - y|^2|x - a|^2 = |y^* - x|^2|y - a|^2.$$




Figure

It follows from (2) and the Figure that if $z \in \partial B$, we have by similar triangles:

(5) $$\frac{|x - a|}{r} = \frac{r}{|x^* - a|} = \frac{|x - z|}{|x^* - z|};$$

namely for any $x \in B$, $x \neq a$:

(6) $$\partial B = \{z \in R^d : \left|\frac{z - x}{z - x^*}\right| = \frac{|x - a|}{r}\}.$$

It is clear that we may put $a = 0$ by the mapping $x \rightsquigarrow x - a$. Next, we put

(7) $$f(x,y) = |x||x^* - y| = |y||y^* - x|,$$

and compute the key formula:

(8) $$f(x,y)^2 = |x|^2 \left|\frac{r^2 x}{|x|^2} - y\right|^2 = r^4 - 2r^2(x,y) + |x|^2|y|^2$$
$$= r^2|x - y|^2 + (r^2 - |x|^2)(r^2 - |y|^2).$$



We now introduce

(9) $$U(x,y) = \frac{A_d}{|x - y|^{d-2}}, \quad (x,y) \in R^d \times R^d$$

with $U(x,x) = +\infty$, where

(10) $$A_d = \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{d/2}}.$$

The function u is known as the Green's function for $R^d$. The Green's function for B is the function G defined on $B \cup (\partial B)$ as follows:

(11) $$G(x,y) = A_d\{(\frac{1}{|x - y|})^{d-2} - (\frac{r}{|x||x^* - y|})^{d-2}\}.$$

Since $|x||x^* - y| > r|x - y|$ by (8), it follows that

(12) $$0 < G(x,y) < U(x,y)$$

in $B \times B$; while

(13) $$G(x,z) = 0$$

on $B \times \partial B$ by (5). For each $y \in B$, it can be verified that $U(\cdot,y) - G(\cdot,y)$ is harmonic in $B - \{y\}$ and takes on the boundary value of $U(\cdot,y)$ on $\partial B$. The last two properties uniquely determine G, and is its raison d'être in classic potential theory. The constant $A_d$ has its significance, but since it plays no role in what follows it is sometimes omitted in the difference of U.

The role of the radius r is not so clear. However, a

straight forward computation shows that if we denote temporarily the G in (11) by $G_r$, we have the reduction formula:

$$G_r(x,y) = \frac{1}{r^{d-2}} G_1\left(\frac{x}{r}, \frac{y}{r}\right). \tag{14}$$

This permits us to concentrate on $B = B(0,1)$ and the G in (11) with $r = 1$. It follows from (4) that G is symmetric in $(x,y)$:

$$G(x,y) = G(y,x).$$

We shall denote the distance of x in B to $\partial B$ by $\delta(x) = 1 - |x|$.

Proposition 1. We have

$$\tfrac{1}{4} \min\left(\frac{1}{|x-y|^{d-2}}, \frac{\delta(x)\delta(y)}{|x-y|^d}\right) < \frac{G(x,y)}{A_d} \tag{15}$$

$$< \min\left(\frac{1}{|x-y|^{d-2}}, \frac{4(d-2)\delta(x)\delta(y)}{|x-y|^d}\right)$$

Proof. The inequality on the right with the first term under min is just (12). Now we write

$$\frac{G(x,y)}{A_d} = \frac{f(x,y)^{d-2} - |x-y|^{d-2}}{|x-y|^{d-2} f(x,y)^{d-2}}. \tag{16}$$

Since $f(x,y) > |x-y|$ by (8) with $r = 1$, the numerator in (16) is less than



$$(d-2)(f(x,y) - |x-y|) f(x,y)^{d-3}$$

$$< (d-2)(f(x,y)^2 - |x-y|^2) f(x,y)^{d-4}$$

$$< 4(d-2)\delta(x)\delta(y) f(x,y)^{d-4}$$

Since $1 - |x|^2 < 2\delta(x)$. Substituting into (16) and using $f(x,y) > |x-y|$ again, we obtain the inequality on the right of (15) with the second term under the min.

On the other hand, the numerator in (16) is greater than

$$(f(x,y) - |x-y|) f(x,y)^{d-3}$$

$$> (f(x,y)^2 - |x-y|^2) \tfrac{1}{2} f(x,y)^{d-4}.$$

Substituting into (16) we obtain

$$\frac{G(x,y)}{A_d} > \frac{f(x,y)^2 - |x-y|^2}{2|x-y|^{d-2} f(x,y)^2}.$$

Since for $A > 0$, $B > 0$, we have $\frac{A}{A+B} > \tfrac{1}{2}\min(1, \frac{A}{B})$, it follows from (8) that

$$\frac{G(x,y)}{A_d} > \frac{1}{4|x-y|^{d-2}} \min\left(1, \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}\right)$$

Since $1 - |x|^2 > \delta(x)$, this implies the left-hand inequality in (15). ∎

Proposition 1 can be blown up as follows. Put

$$(17)\quad F(x,y)=\min\left\{\frac{1}{|x-y|^{d-2}},\frac{\delta(x)}{|x-y|^{d-1}},\frac{\delta(y)}{|x-y|^{d-1}},\frac{\delta(x)\delta(y)}{|x-y|^{d}}\right\}$$

Proposition 2. There exist constants $A_1$ and $A_2$ depending only on d such that for all $(x,y) \in B \times B$:

$$(18)\qquad A_1F(x,y) < G(x,y) < A_2F(x,y)$$

Proof. From here on we shall use A to denote any changeable constant depending only on d. Let us first show that if

$$(19)\qquad G(x,y) < A \min\left(\frac{1}{|x-y|^{d-2}},\frac{\delta(x)\delta(y)}{|x-y|^{d}}\right)$$

then

$$(20)\qquad G(x,y) < 4A\,\frac{\delta(x)}{\delta(y)|x-y|^{d-2}}$$

This is trivial if $2\delta(x) > \delta(y)$. If $\delta(y) > 2\delta(x)$, then

$$|x-y| > \delta(y) - \delta(x) > \tfrac{1}{2}\delta(y)$$

Hence

$$G(x,y) < A\frac{\delta(x)\delta(y)}{|x-y|^{d}} \cdot \frac{4|x-y|^2}{\delta(y)^2} = 4A\frac{\delta(x)}{\delta(y)|x-y|^{d-2}}$$

*Thus (20) is true.* It follows from this and (19) that

$$G(x,y)^2 < A\frac{\delta(x)}{\delta(y)|x-y|^{d-2}}\,\frac{\delta(x)\delta(y)}{|x-y|^{d}}$$



or

$$(21)\qquad G(x,y) < A\frac{\delta(x)}{|x-y|^{d-1}}.$$

This is then also true when $\delta(x)$ is replaced by $\delta(y)$, by the symmetry of G. Hence the right-hand inequality of (18) is true. If we wish we can also insert the right-hand member of (20), and another term obtained from it interchanging x and y, inder the min in (17) for the definition of F. The left-hand inequality then follows automatically from the left-hand inequality in (15).

We now make the important observation that Proposition 2 is invariant when $B(0,1)$ is replaced by $B(0,r)$, provided of course that $\delta(x)$ is interpreted as the distance from x to $\partial B(0,r)$. For if we write this distance as $\delta_r(x)$, then $\delta_r(x) = r - |x| = r\delta_1(\frac{x}{r})$, so that

$$(22)\qquad F(\tfrac{x}{r},\tfrac{y}{r}) = r^{d-2}F(x,y).$$

Therefore by (14), the inequalities in (18) are unchanged when $B(0,1)$ is replaced by $B(0,r)$. Similarly, the constant A in the next proposition does not depend on r. The next result originated with Brossard.

Proposition 3. There exists a constant A depending only on d such that

$$(23)\qquad \frac{G(x,y)G(y,z)}{G(x,z)} < A\frac{U(x,y)U(y,z)}{U(x,z)}$$

for all x, y and z in B.

Proof. We have by (18) and (20):

$$G(x,y)G(y,z) < A\frac{\delta(x)\delta(z)}{|x - y|^{d}|y - z|^{d-2}};$$

hence also by symmetry

$$G(x,y)G(y,z) < A\frac{1}{|x-y|^{d-2}|y-z|^{d-2}}\min\left\{1, \frac{\delta(x)\delta(z)}{|x-y|^2}, \frac{\delta(x)\delta(z)}{|y-z|^2}\right\}.$$

On the other hand we have by (15):

$$G(x,z) > \frac{A}{|x - z|^{d-2}}\min\left\{1, \frac{\delta(x)\delta(z)}{|x - z|^2}\right\}.$$

Hence if $\delta(x)\delta(z) > |x - z|^2$, the left member of (23) does not exceed

$$\frac{1}{A}\left|\frac{x - z}{(x - y)(y - z)}\right|^{d-2};$$

if $\delta(x)\delta(z) < |x - z|^2$, it does not exceed

$$A\left|\frac{x - z}{(x - y)(y - z)}\right|^{d-2}\min\left\{\left|\frac{x - z}{x - y}\right|^2, \left|\frac{x - z}{y - z}\right|^2\right\}.$$

Since $|x - y| + |y - z| > |x - z|$, the last-written min does not exceed 4. This establishes (23).

Let $w \in \partial B$, then

$$\lim_{z \to w}\frac{G(x,z)}{\delta(z)} = -\frac{\partial}{\partial n_w}G(x,w) = K(x,w)$$

where $\frac{\partial}{\partial n_w}$ denotes the outward normal derivative at $w$, since $G(x,w) = 0$ for $x \in B$, by (13). The function $K(\cdot,\cdot)$ defined on $B \times \partial B$ is known as Poisson's kernel. Dividing



the left member of (23) by $\delta(z)$ in both numerator and denominator, and letting $z \to w$, we obtain

(24) $$G^{w}(x,y) \overset{\Delta}{=} \frac{G(x,y)K(y,w)}{K(x,w)} < C\frac{U(x,y)U(y,w)}{U(x,w)}.$$

However, for the ball $B(0,r)$, Poisson's kernel is known explicitly:

(25) $$K(x,z) = \frac{A_d d}{r}\,\frac{r^2 - |x|^2}{|x - z|^d}$$

where $A_d$ is given by (10). Hence (24) is trivial. Prom (24) we derive easily the inequality

(26) $$G^{w}(x,y) < A\max\left(\frac{1}{|x - y|^{d-2}}, \frac{1}{|y - w|^{d-2}}\right)$$

which is a fundamental estimate, also given by Brossard. His proof is quite different.

We now consider $B = B(0,r)$ in $R^2$. In this case the Green's function for $B$ is given by

(27) $$G(x,y) = \frac{1}{\pi}\log\frac{|x|\,|x^* - y|}{r\,|x - y|},\quad (x,y) \in B \times B.$$

Then $G(x,y) > 0$, and $= 0$ if $x \in \partial B$ or $y \in \partial B$, as before. We put

(28) $$U_r(x,y) = \log\frac{3r}{|x - y|}$$

so that $U_r > \log\frac{3}{2} > \frac{2}{5}$ in $\overline{B} \times \overline{B}$. Put also

(29) $$\phi(x,y) = \frac{|x|\,|x^* - y|}{r}.$$

Then by (8),

(30) $\phi(x,y)^2 = |x - y|^2 + \frac{1}{r^2}(r^2 - |x|^2)(r^2 - |y|^2).$

Hence $\phi^2 < (2r)^2 + r^2 = 5r^2$. Now we represent G as follows

(31) $$G(x,y) = \frac{1}{\pi}\{U_r(x,y) + \log \frac{\phi(x,y)}{3r}\}$$

The second term in the right member above is negative because $\sqrt{5}/3 < 1$, whereas the first term is bounded away from zero. This explains our choice of 3r in (28) rather than the usual one. An immediate consequence is that

(32) $$G(x,y) < \frac{1}{\pi}U_r(x,y).$$

Next, we have from (30)

(33) $\log \phi(x,y) = \log |x - y| + \frac{1}{2} \log (1 + \psi(x,y))$

where

(34) $$\psi(x,y) = \frac{(r^2 - |x|^2)(r^2 - |y|^2)}{r^2|x - y|^2}.$$

Since $r^2 - |x|^2 < 2r\delta(x)$, we have

(35) $$\psi(x,y) < \frac{4\delta(x)\delta(y)}{|x - y|^2}.$$

Since $\psi > 0$, $\log(1 + \psi) < \psi$; it follows from (33) and (35) that



(36) $$\log \phi(x,y) < \log |x - y| + \frac{2\delta(x)\delta(y)}{|x - y|^2}.$$

Observing that

(37) $$G(x,y) = \frac{1}{\pi} \log \frac{\phi(x,y)}{|x - y|}$$

we obtain from (36) that

(38) $$G(x,y) < \frac{2}{\pi} \frac{\delta(x)\delta(y)}{|x - y|^2}.$$

Continuing (32) and (38), we have (using $a \wedge b$ to denote min (a,b)):

(39) $$G(x,y) < \frac{1}{\pi}\{U_r(x,y) \wedge \frac{2\delta(x)\delta(y)}{|x - y|^2}\}.$$

Since $U_r > \frac{2}{5}$, this leads to the next proposition.

Proposition 4. In $R^2$, the Green's function. G for B(0,r) satisfies the following inequality:

(40) $$G(x,y) < \frac{1}{\pi} \log \frac{3r}{|x - y|}\{1 \wedge \frac{5\delta(x)\delta(y)}{|x - y|^2}\}$$

In contrast to Proposition 1 in the case $d > 3$, the inequality (40) cannot be reversed by changing the constants involved. In other words, there does not exist any constant $A > 0$ such that

(41) $$G(x,y) > A \log \frac{3r}{|x - y|}\{1 \wedge \frac{\delta(x)\delta(y)}{|x - y|^2}\}.$$

To see this let $0 < \varepsilon < 1$ and $\delta(x) = \delta(y) = |x - y| = \varepsilon r$. By (38), we have $G(x,y) \le \frac{2}{\pi}$, whereas the right member of (41) reduces to $A \log \frac{3}{\varepsilon}$. It is not clear whether there exists a "sharp" estimate for G as in the case $d \ge 3$ above.

We proceed to an analogue for (26). For $d = 2$, the analogue of (25) is given by

$$(42) \qquad K(x,w) = \frac{1}{2\pi r}\,\frac{r^2 - |x|^2}{|x - w|^2}, \quad (x,w) \in B \times \partial B;$$

while $G^w(x,y)$ is defined as in (24). We have then

$$(43) \qquad G^w(x,y) \le G(x,y)\frac{2\delta(y)}{\delta(x)}\,\frac{|x - w|^2}{|y - w|^2}$$

because $r^2 - |y|^2 \le 2r\delta(y)$, $r^2 - |x|^2 \ge r\delta(x)$. Observe that

$$(44) \qquad 1 \wedge \frac{5\delta(x)\delta(y)}{|x - y|^2} \le \frac{7\delta(x)}{\delta(y)}$$

This is trivial if $\delta(y) \le 7\delta(x)$; otherwise it follows from $|x - y| \ge \delta(y) - \delta(x) \ge \frac{6}{7}\delta(y)$ and $5(\frac{7}{6})^2 \le 7$. Using (44) in (40), we obtain

$$(45) \qquad G(x,y) \le \frac{7}{\pi}\log\frac{3r}{|x - y|}\left\{\frac{\delta(x)}{\delta(y)} \wedge \frac{\delta(x)\delta(y)}{|x - y|^2}\right\}.$$

Therefore we have by (43)

$$(46) \qquad G^w(x,y) \le \frac{14}{\pi}\log\frac{3r}{|x - y|}\left\{1 \wedge \frac{\delta(y)^2}{|x - y|^2}\right\}\frac{|x - w|^2}{|y - w|^2}.$$

$$\le \frac{14}{\pi}\log\frac{3r}{|x - y|}\left\{\frac{|x - w|^2}{|y - w|^2} \wedge \frac{|x - w|^2}{|x - y|^2}\right\}$$



because $\delta(y) \le |y - w|$. The quantity between the braces above does not exceed 4, as shown in the proof of Proposition 3.

Proposition 5. For any $w \in \partial B$, we have

$$(47) \qquad G^w(x,y) \le \frac{56}{\pi}\log\frac{3r}{|x - y|}.$$

In contrast to (26), this estimate of $G^w$ does not depend on w.

Postscript. Some of the results above are implicit in the work by Z. Zhao, but the arrangements as well as formulations may be new. For instance, experts we consulted were not aware of the sharp form given in Proposition 2. It has since been proved for a bounded $C^{1,1}$ domain in $R^d$, $d \ge 3$, by Zhao (to appear in a book by us). According to some experts, once the results are established for a ball, geometrical transformations yield easily their extensions to a "reasonably smooth" domain. Although I am not privy to such arguments, this consideration makes it worthwhile to examine the case of a ball in detail.

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